JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 24, 296–306 (1968)

Regularity of Growth and Gaps

L. R. SONS

Northern Illinois University Dekalb

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $f(z)$ be an entire function, and let $M(r) = M(r, f)$ be the maximum of $f(z)$ on $|z| = r$. Set

$$\overline{\underline{\lim}}_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \begin{cases}\rho_0 \\ \lambda_0\end{cases}.$$

ρ_0 and λ_0 are called the order and lower order, respectively, of the function f. The function f is said to be of regular growth if $\rho_0 = \lambda_0$.

If

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k},$$

where the $\{n_k\}$ are those integers for which $c_k \neq 0$, Valiron [8] noted that $\lambda_0 = \rho_0 > 0$ implies

$$\underline{\lim}_{k\to\infty} \frac{\log n_k}{\log n_{k+1}} = 1.$$

More precisely, Whittaker [9] showed

$$\lambda_0 \leqslant \rho_0 \underline{\lim}_{k\to\infty} \frac{\log n_k}{\log n_{k+1}}.$$

We shall obtain the relation corresponding to Whittaker's result for functions analytic in $|z| < 1$.

If $f(z)$ is analytic in $|z| < 1$, we define

$$\overline{\underline{\lim}}_{r\to 1} \frac{\log^+\log^+ M(r,f)}{-\log(1-r)} = \begin{cases}\rho^* \\ \lambda^*\end{cases}.$$

ρ^* and λ^* are called the M-order and lower M-order respectively of the function f. The function f is said to be of regular growth with respect to M-order if $\rho^* = \lambda^*$. Using these definitions we have the following result:

THEOREM 1. *If*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \tag{1}$$

is analytic in $|z| < 1$, *and if* $0 < \rho^* < \infty$, *then*

$$1 + \lambda^* \leqslant (1 + \rho^*) \varliminf_{k\to\infty} \frac{\log n_k}{\log n_{k+1}}.$$

In particular, if $\lambda^* = \rho^*$, *then*

$$\varliminf_{k\to\infty} \frac{\log n_k}{\log n_{k+1}} = 1.$$

Furthermore, if $f(z)$ is not of regular growth with respect to M-order, we have

THEOREM 2. *If* $f(z)$ *is analytic in* $|z| < 1$ *with* $\rho^* > 0$ *and* $\lambda^* < \mu < \rho^*$, *then*

$$f(z) = g(z) + h(z)$$

where $g(z)$ *has M-order less than or equal to* μ *and*

$$h(z) = \sum_{k=0}^{\infty} b_k z^{m_k}$$

($\{m_k\}$*) the integers for which* $b_k \neq 0$*) satisfies*

$$1 + \lambda^* \geqslant (1 + \mu) \varliminf_{k\to\infty} \frac{\log m_k}{\log m_{k+1}}.$$

If $T(r) = T(r, f)$ is the Nevanlinna characteristic of the entire function f (see, for example, Hayman [1]), then it is easily shown using the inequality

$$T(r) \leqslant \log^+ M(r) \leqslant \frac{R + r}{R - r}\, T(R), \qquad (0 \leqslant r < R), \tag{2}$$

that

$$\frac{\varlimsup}{\varliminf}_{r\to\infty} \frac{\log T(r, f)}{\log r} = \begin{cases}\rho_0 \\ \lambda_0\end{cases}.$$

For a function f which is analytic in $|z| < 1$ we let

$$\overline{\lim_{r\to 1}} \frac{\log T(r,f)}{-\log(1-r)} = \begin{cases}\rho \\ \lambda\end{cases}.$$

ρ and λ are called the T-order and lower T-order, respectively, of f. The function f is said to be of regular growth with respect to T-order if $\rho = \lambda$.

ρ and ρ^* need not be equal. In fact, for $f(z) = \exp((1+z)/(1-z))$, it is easy to see $\lambda^* = \rho^* = 1$, but $\lambda = \rho = 0$. Also we have

THEOREM 3. *There exist functions $f(z)$ analytic in $|z| < 1$ for which $\lambda^* = \rho^*$, but $\lambda \neq \rho$.*

(I am grateful to Prof. W. H. J. Fuchs for bringing the example given here to my attention.)

However, it follows from (2) that

$$\rho \leqslant \rho^* \leqslant \rho + 1, \tag{3}$$

and

$$\lambda \leqslant \lambda^* \leqslant \lambda + 1. \tag{4}$$

Hence we can say:

(a) If $\rho = \rho^*$ and $\rho = \lambda$, then $\rho^* = \lambda^*$.

(b) If $\lambda = \lambda^*$ and $\rho^* = \lambda^*$, then $\rho = \lambda$.

As a corollary to Theorem 1 we then have

THEOREM 4. *Let $f(z)$ be analytic in $|z| < 1$ with order $0 < \rho^* < \infty$ and have the form* (1). *If $\lambda = \rho = \rho^*$, then*

$$\underline{\lim_{k\to\infty}} \frac{\log n_k}{\log n_{k+1}} = 1.$$

The example $f(z) = \exp((1+z)/(1-z))$ shows that we may have $\lambda = \rho$ and $\lambda^* = \rho^*$ without $\rho = \rho^*$ (or $\lambda = \lambda^*$). For $\rho < \rho^*$ we can say

THEOREM 5. *Let $f(z)$ be analytic in $|z| < 1$ and have the form* (1). *If $\rho < \rho^* < \infty$, then*

$$\overline{\lim_{k\to\infty}} \frac{\log n_k}{\log n_{k+1}} = 1, \tag{5}$$

and

$$\underline{\lim_{k\to\infty}} \frac{\log n_k}{\log n_{k+1}} \geqslant \frac{1+\lambda}{2+\rho}. \tag{6}$$

Line (6) follows from Theorem 1 and the inequalities (3) and (4). Line (5) yields another class of functions analytic in $|z| < 1$ for which $\rho = \rho^*$ (see also Shea [5] and Linden ([2] and [3])).

COROLLARY. *If $f(z)$ is analytic in $|z| < 1$ and has the form* (1) *with*

$$\overline{\lim_{k\to\infty}} \frac{\log n_k}{\log n_{k+1}} < 1,$$

then $\rho = \rho^*$.

Finally we remark that Theorems 1, 4, and 5 enable us to easily construct a function analytic in $|z| < 1$ which has neither M-order regularity nor T-order regularity.

2. PROOF OF THEOREM 1

For $f(z)$ analytic in $|z| < 1$ we shall denote the maximum term on $|z| = r$ of the power series expansion of $f(z)$ about zero by $\mu(r)$ and the rank of the maximum term by $\nu(r)$. We need four lemmas.

LEMMA 1 (Valiron [7]). *If f is analytic in $|z| < 1$ and $\rho^* > 0$, then*

$$M(r) < \mu(r)\left[2\nu\left(r + \frac{1-r}{\nu(r)}\right) + 1\right] \cdot \frac{1}{1-r}, \qquad (r > r_0), \tag{7}$$

$$\log \mu(r) = \log \mu(r') + \int_{r'}^{r} \frac{\nu(x)}{x}\, dx, \qquad (r, r' > r_1). \tag{8}$$

LEMMA 2. *If f is analytic in $|z| < 1$ and $0 < \rho^* < \infty$, then*

$$\overline{\lim_{r\to 1}} \frac{\log \nu(r)}{-\log(1-r)} = 1 + \rho^*. \tag{9}$$

Also,

$$\overline{\underline{\lim}}_{r\to 1} \frac{\log\log \mu(r)}{-\log(1-r)} = \begin{Bmatrix}\rho^* \\ \lambda^*\end{Bmatrix}. \tag{10}$$

PROOF OF LEMMA 2. If $f(z) = \sum c_k z^k$, MacLane [4] has shown that

$$\overline{\lim_{k\to\infty}} \frac{\log^+ \log^+ |c_k|}{\log k - \log^+ \log^+ |c_k|} = \rho^*.$$

Hence, for each $\epsilon > 0$, there is a value r_0 such that

$$\log \mu(r) \leqslant \nu(r)^{(\rho^*/)1+\rho^*)+\epsilon)}, \qquad \text{for} \qquad r > r_0. \tag{11}$$

Therefore,

$$\rho^* \leqslant \frac{\rho^*}{1+\rho^*} \overline{\lim_{r\to 1}} \frac{\log \nu(r)}{-\log(1-r)}.$$

For $\epsilon > 0$, we also know

$$\log \mu(r) \leqslant \log M(r) \leqslant (1-r)^{-(\rho^*+\epsilon)}, \qquad (r > r_1).$$

Thus using (8) we find for $r > r_2$

$$\nu(r)\, A(1-r) \leqslant \nu(r) \log\left(\frac{r + \frac{1}{2}(1-r)}{r}\right) \leqslant \int_r^{r+\frac{1}{2}(1-r)} \frac{\nu(t)}{t}\, dt$$

$$\leqslant B(1-r)^{-(\rho^*+\epsilon)},$$

where A and B are positive constants. The extremities of this inequality now yield (9).

Since $\log \mu(r) \leqslant \log M(r)$, we know

$$\overline{\lim_{r\to 1}} \frac{\log\log \mu(r)}{-\log(1-r)} \leqslant \begin{cases} \rho^* \\ \lambda^* \end{cases}.$$

The opposite inequalities follow in a straightforward manner from (7), (9), and the fact that for $\rho^* > 0$, we know $\nu(r) \to \infty$ and $\mu(r) \to \infty$ as $r \to 1$.

LEMMA 3. *Let $\varphi(x)$ be a positive, nondecreasing function on* (0, 1) *such that*

$$\lim_{x\to 1} \frac{\log \varphi(x)}{-\log(1-x)} = \alpha.$$

Let β, γ be real numbers for which $\beta > \alpha$ and $\alpha/\beta < \gamma < 1$. If x' is a number for which $\frac{1}{2} \leqslant r(\gamma) < x' < 1$ and also

$$\frac{\log \varphi(x')}{-\log(1-x')} \leqslant \beta\gamma,$$

then for $2x' - 1 \leqslant x \leqslant x'$,

$$\varphi(x) \leqslant (1-x)^{-\beta}.$$

PROOF OF LEMMA 3. We see that

$$\log \varphi(x) \leqslant \log \varphi(x') \leqslant \beta\gamma(-\log(1-x')).$$

But

$$(1 - x')^{-\gamma} \leqslant (1 - (2x' - 1))^{-1} \leqslant (1 - x)^{-1}$$

provided

$$\log (1 - (2x' - 1)) = \log 2(1 - x') \leqslant \frac{-\gamma \log 2}{1 - \gamma}.$$

LEMMA 4. *If f is analytic in $|z| < 1$ with $0 < \rho^* < \infty$, then*

$$\varliminf_{r \to 1} \frac{\log \nu(r)}{-\log (1 - r)} = 1 + \varliminf_{r \to 1} \frac{\log \log \mu(r)}{-\log (1 - r)} = 1 + \lambda^*. \tag{12}$$

PROOF OF LEMMA 4. From (8) we conclude for $r > r_0$,

$$\nu(r)\, A(1 - r) \leqslant \int_r^{r + \frac{1}{2}(1-r)} \frac{\nu(t)}{t}\, dt \leqslant \log \mu(r + \tfrac{1}{2}(1 - r)),$$

where A is a positive constant. Thus, since for each r in $(0, 1)$ $\log \mu(r) \leqslant \log M(r)$,

$$\alpha = \varliminf_{r \to 1} \frac{\log \nu(r)}{-\log (1 - r)} \leqslant 1 + \lambda^*.$$

To show $\alpha \geqslant 1 + \lambda^*$ we first note that (11) yields

$$\lambda^* \leqslant \frac{\rho}{1 + \rho^*} \cdot \alpha \tag{13}$$

and also for $r > r^*$

$$\log \mu(r) < \nu(r). \tag{14}$$

Let ϵ be a small positive number for which $\alpha + \epsilon \neq 1$, and let S_1 be the set of all maximal (in length) intervals of the form $[\rho_i, \rho_i']$ for which

$$\log \mu(\rho) \leqslant (1 - \rho)^{-(\lambda^* + \frac{1}{2}\epsilon)} \tag{15}$$

when $\rho_i \leqslant \rho \leqslant \rho_i'$. Thus, every ρ in $(0, 1)$ for which (15) holds is in precisely one interval in the set S_1. From Lemma 3 it is clear that each interval $[\rho_i, \rho_i']$ in S_1 *contains* the associated interval $[2\rho_i'^{-1}, \rho_i']$ provided $\rho_i' > r(\epsilon)$.

Now let S_2 be the set of all maximal (in length) intervals of the form $[r_i, r_i')$ for which

$$\nu(r) \leqslant (1 - r)^{-(\alpha + \epsilon)}$$

when $r_i \leqslant r < r_i'$. Since (14) holds, when $r_i > r^*$ we have also for $r_i \leqslant r < r_i'$ that

$$\log \mu(r) < (1 - r)^{-(\alpha+\epsilon)}.$$

But by (13) we know $\alpha \geqslant \lambda^*$, so $\alpha + \epsilon > \lambda^* + \frac{1}{2}\epsilon$. Hence, each $[\rho_i, \rho_i']$ above with $\rho_i > r^*$ is contained in some $[r_j, r_j')$.

Therefore, using (8) for $[\rho_i, \rho_i']$ with $\rho_i > \max(r^*, r(\epsilon))$ we find

$$\begin{aligned}
\log \mu(\rho_i') - \log \mu(2\rho_i' - 1) &= \int_{(2\rho_i'-1)}^{\rho_i'} \frac{\nu(t)}{t}\, dt \\
&\leqslant \int_{(2\rho_i'-1)}^{\rho_i'} \left(\frac{1}{1-t}\right)^{\alpha+\epsilon} \frac{1}{t}\, dt \\
&\leqslant \left(\frac{1}{2\rho_i' - 1}\right)\left(\frac{1}{1-(\alpha+\epsilon)}\right) \\
&\quad \times \left(\left(\frac{1}{1-(2\rho_i'-1)}\right)^{\alpha+\epsilon-1} - \left(\frac{1}{1-\rho_i'}\right)^{\alpha+\epsilon-1}\right).
\end{aligned} \tag{16}$$

However, the properties of S_1, and in particular (15), give

$$\begin{aligned}
\log \mu(\rho_i') - \log \mu(2\rho_i' - 1) &\geqslant (1-\rho_i')^{-(\lambda^*+\frac{1}{2}\epsilon)} - (1-(2\rho_i'-1))^{-(\lambda^*+\frac{1}{2}\epsilon)} \\
&\geqslant (1-\rho_i')^{-(\lambda^*+\frac{1}{2}\epsilon)} \cdot (1 - 2^{-(\lambda^*+\frac{1}{2}\epsilon)}).
\end{aligned} \tag{17}$$

Combining (16) and (17), we see for $\rho_i > \max(r^*, r(\epsilon))$,

$$(1-\rho_i')^{-(\lambda^*+\frac{1}{2}\epsilon)} \cdot K \leqslant (\alpha+\epsilon-1)^{-1}((1-\rho_i')^{-(\alpha+\epsilon-1)} - (2(1-\rho_i')^{-(\alpha+\epsilon-1)}), \tag{18}$$

where $K = (1 - 2^{-(\lambda^*+\frac{1}{2}\epsilon)})(2\rho_i' - 1)$. We note immediately that $\alpha \geqslant 1$, since $\lambda^* \geqslant 0$, ϵ can be arbitrarily small, and there must be ρ_i' arbitrarily close to one for which (18) holds. Again using the fact that there must be ρ_i arbitrarily close to one for which (18) holds, we then conclude that $\lambda^* + \frac{1}{2}\epsilon \leqslant \alpha + \epsilon - 1$ for arbitrary $\epsilon > 0$. Thus, $\lambda^* + 1 \leqslant \alpha$.

COMPLETION OF THE PROOF OF THEOREM. Let

$$\alpha = \varlimsup_{k\to\infty} \frac{\log n_k}{\log n_{k+1}}.$$

If $\beta > \alpha$, there exists a sequence $\{p(k)\}$ such that $n_{p(k)} < n^{\beta}_{p(k)+1}$. Let r_t be a value of r at which $\nu(r)$ jumps from a value less than or equal to $n_{p(t)}$ to a value greater than or equal to $n_{p(t)+1}$. Then Lemmas 2 and 4 show

$$\lambda^* + 1 \leqslant \varlimsup_{t\to\infty} \frac{\log \nu(r_t - 0)}{\log r_t} \leqslant \beta \varlimsup_{t\to\infty} \frac{\log \nu(r_t + 0)}{\log r_t} \leqslant \beta(\rho^* + 1).$$

3. Proof of Theorem 2

If $f(z) = \sum c_k z^k$, MacLane [4] has shown that

$$\varlimsup_{k\to\infty} \frac{\log^+ \log^+ |c_k|}{\log k - \log^+ \log^+ |c_k|} = \rho^*. \tag{19}$$

Let $g(z) = \sum a_k z^k$ where $a_k = c_k$ if

$$\log^+ |c_k| \leqslant k^{[\mu/(1+\mu)]};$$

otherwise $a_k = 0$. Then (as in (19)) g has order less than or equal to μ. Set

$$h(z) = f(z) - g(z) = \sum a_{m_k} z^{m_k}$$

with

$$A_{m_k} = |a_{m_k}| \qquad \text{and} \qquad \log^+ A_{m_k} > (m_k)^{[\mu/(1+\mu)]}.$$

Let

$$r_k = 1 - e^{-1}(m_k)^{-1/(\mu+1)}, \qquad (k = 1, 2, \ldots).$$

Thus, for $r_k \leqslant r \leqslant r_{k+1}$,

$$M(r) \geqslant A_{m_k} r^{m_k} \geqslant A_{m_k} r_k^{m_k},$$

and

$$\frac{\log\log M(r)}{-\log(1-r)} \geqslant \frac{\log\{\log A_{m_k} + m_k \log r_k\}}{-\log(1 - r_{k+1})} \geqslant \frac{\log\{m_k^{\mu/(1+\mu)} + m_k \log r_k\}}{1 + \left(\frac{1}{\mu+1}\right)\log m_{k+1}}$$

$$\geqslant \frac{(\mu+1)\log m_k}{(\log m_{k+1})\left(1 + \frac{\mu+1}{\log m_{k+1}}\right)}$$

$$+ \frac{(\mu+1)\log\left\{m_k^{-1/(\mu+1)} + \log\left(1 - \frac{1}{e}\left(\frac{1}{m_k}\right)^{1/(\mu+1)}\right)\right\}}{\log m_{k+1} + \mu + 1}. \tag{20}$$

The power series expansion for $\log(1 - x)$ enables one to see easily for $k \geqslant k_0$,

$$\left(\frac{1}{m_k}\right)^{1/(\mu+1)} + \log\left(1 - \frac{1}{e}\left(\frac{1}{m_k}\right)^{1/(\mu+1)}\right) \geqslant \left(\frac{1}{m_k}\right)^{1/(\mu+1)} \cdot \left(1 - \frac{5}{4e}\right)$$

$$\leqslant \left(\frac{1}{m_{k+1}}\right)^{1/(\mu+1)} \cdot \left(1 - \frac{5}{4e}\right).$$

Thus, (20) gives

$$\lambda^* = \underline{\lim}_{r\to 1} \frac{\log\log M(r)}{-\log(1-r)} \geqslant (\mu+1)\, \underline{\lim}_{k\to\infty} \frac{\log m_k}{\log m_{k+1}} - 1.$$

4. Proof of Theorem 3

Let ρ_1 be a positive real number and $\lambda_1 \geqslant 0$ such that $\lambda_1 < \rho_1$. Using Shea [5] we can find a function $g(z)$ analytic in $|z| < 1$ for which $g(r) = M(r, g)$ and

$$T(r, g) \sim \log M(r, g) \sim (1 - r)^{-\alpha(r)},$$

where $\alpha(r)$ is a real-valued function whose values vary between λ_1 and ρ_1 and which comes arbitrarily close to each of these values as r approaches one.

Set

$$E(z) = \exp\{(1 - z)^{-\rho_1}\} \qquad \text{and} \qquad f(z) = E(z) + g(z).$$

Since the maximum of $|E(z)|$ on $|z| = r$ is assumed on the positive real axis, we have

$$M(r, f) = E(r) + g(r) = M(r, E) + M(r, g).$$

Thus,

$$\lim_{r\to 1} \frac{\log\log M(r, f)}{-\log(1-r)} = \rho_1 . \tag{21}$$

Turning to T-order, we see as in Tsuji [6, p. 205] that

$$\overline{\lim_{r\to 1}} \frac{\log T(r, E)}{-\log(1-r)} = \rho_1 - 1.$$

A standard inequality for the Nevalinna characteristic shows

$$T(r, f) \leqslant T(r, g) + T(r, E) + \log 2.$$

Hence, since there exists a sequence of r values approaching one for which $\alpha(r)$ approaches λ_1, we find

$$\varliminf_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} \leqslant \gamma, \tag{22}$$

where

$$\gamma = \max(\lambda_1, \rho_1 - 1).$$

It is also true that

$$T(r,g) \leqslant T(r,f) + T(r,E) + \log 2,$$

and thus if r approaches one through a sequence of values for which $\alpha(r)$ approaches ρ_1, we find

$$\varlimsup_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} \geqslant \rho_1 . \tag{23}$$

Comparing (21), (22), and (23), we have the desired result. (Incidentally, (3) tells us $\rho = \rho^* = \rho_1$).

5. Proof of Theorem 5

Suppose

$$\varlimsup_{k\to\infty} \frac{\log n_k}{\log n_{k+1}} < 1.$$

Then there is a number q such that

$$\frac{n_{k+1}}{n_k} \geqslant q > 1, \qquad (0, 1, 2,\ldots).$$

Applying Zygmund [10, p. 216], we find positive numbers λ_q and μ_q depending only on q such that

$$|f(re^{i\theta})| > \lambda_q \left(\sum_{k=0}^{\infty} |c_k|^2 r^{2n_k}\right)^{1/2}$$

on a set of θ of measure not less than μ_q. Therefore,

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta > \mu_q \left(\tfrac{1}{2}\log\left(\sum_{k=0}^{\infty} |c_k|^2 r^{2n_k}\right) + \log \lambda_q\right)$$

$$\geqslant \mu_q \log \mu(r).$$

By (10) of Lemma 2 for $0 < \epsilon < \frac{1}{3}(\rho^* - \rho)$ there is a sequence of r values tending to one for which

$$\log \mu(r) > (1 - r)^{-(\rho^* - \epsilon)}.$$

For such r values

$$T(r) = m(r, f) > \mu_q(1 - r)^{-(\rho^* - \epsilon)}. \tag{24}$$

However, since f has T-order ρ, for $r > r_0$,

$$T(r) < (1 - r)^{-(\rho + \epsilon)}. \tag{25}$$

Thus, (24) and (25) hold for a sequence of r values tending to one which contradicts $\rho < \rho^*$. So (5) is verified.

References

1. W. K. Hayman. "Meromorphic Functions." Oxford, London, 1964.
2. C. N. Linden. On a conjecture of Valiron concerning sets of indirect Borel points. *J. London Math. Soc.* **41** (1966), 304–312.
3. C. N. Linden. The representation of regular functions. *J. London Math. Soc.* **39** (1964), 19–30.
4. G. R. MacLane. "Asymptotic Values of Holomorphic Functions. "Rice University Studies (Houston, 1963).
5. D. F. Shea. Functions analytic in a finite disk and having asymptotically prescribed characteristic. *Pacific J. Math.* **17** (1966), 549–560.
6. M. Tsuji. "Potential Theory in Modern Function Theory." Maruzen, Tokyo, 1959.
7. G. Valiron. "Fonctions analytiques." Presses Universitaires de France, Paris, 1954.
8. G. Valiron. "Lectures on Integral Functions." Chelsea, New York 1949.
9. J. M. Whittaker. The lower order of integral functions. *J. London Math. Soc.* **8** (1933), 20–27.
10. A. Zygmund. "Trigonometric Series," Second Ed., Vol. I. Cambridge, 1959.